

PERCOLATION OF HARD DISKS

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ABSTRACT. Random arrangements of points in the plane, interacting only through a simple hard core exclusion, are considered. An intensity parameter controls the average density of arrangements, in analogy with the Poisson point process. It is proved that at high intensity, an infinite connected cluster of excluded volume appears with positive probability.

1. INTRODUCTION

Consider a random arrangement of points in the plane. Suppose that each pair of points at distance less than L from one another are joined by an edge, and let G be the resulting graph. An important question in percolation theory is: Does G have an infinite connected component?

A key problem in answering this question is in defining what is meant by a random arrangement of points. A standard model is the *Poisson point process*, in which the probability that a (Borel) set A contains k points of the random arrangement is Poisson distributed with parameter $\lambda|A|$, where $|\cdot|$ is Lebesgue measure and λ is the *intensity* of the process. Events in disjoint sets are independent [1]. Here λ is the (average) density of arrangements of points; it can be shown that if λ is greater than some critical value λ_c , then G has an infinite connected component with probability one [2]. (Of course λ_c depends on the connection distance L .)

The Poisson point process is closely related to the (grand canonical) *Gibbs distribution* of statistical mechanics (with particle interaction set to zero and momentum variables integrated out) in the sense that they give nearly identical probabilistic descriptions of arrangements of points in large finite subsets of the plane. The Gibbs distributions, however, also allow for interactions among the points. Suppose the points interact through a simple exclusion of radius $2r > 0$. (That is, each pair of points is separated by a distance of at least $2r$.) Each arrangement of points can then be imagined as a collection of *hard core* (i.e., nonoverlapping) disks of radius r .

There is a Gibbs distribution on arrangements of points with exclusion radius $2r$ in finite subsets of the plane which, like the Poisson process, gives equal probabilistic weight to every arrangement of the same density. Furthermore a probability measure can be defined on such arrangements in the whole plane, such that in a certain sense its restriction

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to finite subsets has the Gibbs distribution. This probability measure, called an (infinite volume) *Gibbs measure*, has been extensively studied (see e.g. [4],[5],[6]).

It is natural to ask whether G has an infinite connected component when the points in G are sampled from a Gibbs measure with an exclusion of radius $2r$. If $r \ll L$, one can argue that the exclusion is insignificant and that, by analogy with the Poisson process, there is some critical *activity*, z_c , such that G almost surely has an infinite connected component for $z > z_c$. (See Section 7 of [7] for a sketch of a proof in this direction.) Here the activity z is a parameter analogous to the intensity of the Poisson process.

If r and L are close the qualitative relationship with the Poisson point process is less clear, at least as it pertains to percolation. In particular, let $L < 4r$. Then the percolation question is closely related to *excluded volume*. (The excluded volume corresponding to an arrangement of points is the set of all points which, due to the exclusion radius, cannot be added to the arrangement.) If G has an infinite component for such L , then there is an infinite connected region of excluded volume. The latter event has been associated with the gas/liquid phase transition in equilibrium statistical mechanics [8],[9]. Below it is proved that given $L > 3r$, with points distributed under a Gibbs measure with an exclusion of radius $2r$, G has an infinite connected component with positive probability whenever the activity z is sufficiently large.

Little is known about qualitative properties of typical samples from a Gibbs measure (with exclusion) when z is large; even simulations have been inconclusive, although a recent large-scale study [10] may settle some questions. It is expected (but not proven) that when z is large, typical arrangements exhibit long-range orientational order [10]. On the other hand, it has been shown that there can be no long-range positional order at any z (see [11]; this is an extension of the famous Mermin-Wagner theorem to the case of hard core interactions). The absence of long-range positional order makes the percolation question even more pertinent.

2. NOTATION, PROBABILITY MEASURE, AND SKETCH OF PROOF

Fix $r > 0$, and define

$$\Omega = \{\omega \subset \mathbb{R}^2 : |x - y| \geq 2r \ \forall \ x \neq y \in \omega\} \subset \mathcal{P}(\mathbb{R}^2)$$

In particular $\emptyset \in \Omega$. (Here $\mathcal{P}(\mathbb{R}^2)$ is the set of subsets of \mathbb{R}^2 .) Let \mathcal{T} be the topology on Ω generated by the subbasis of sets of the form

$$\{\omega \in \Omega : \#(\omega \cap U) = \#(\omega \cap K) = m\}$$

for compact sets $K \subset \mathbb{R}^2$, open sets $U \subset K$, and positive integers m . Here $\#\zeta$ is the number of elements in the set ζ . Let \mathcal{F} be the σ -algebra of Borel sets with respect to the topology \mathcal{T} . It can be shown that \mathcal{F} is generated by sets of the form

$$\{\omega \in \Omega : \#(\omega \cap B) = m\}$$

for bounded Borel sets $B \subset \mathbb{R}^2$ and nonnegative integers m [5]. Let

$$\Lambda_n = [-n, n]^2 \subset \mathbb{R}^2$$

and given $A \in \mathcal{F}$, define

$$A_{n,N} = \{(x_1, \dots, x_N) : \{x_1, \dots, x_N\} \in A, \{x_1, \dots, x_N\} \subset \Lambda_n\} \subset (\mathbb{R}^2)^N$$

and

$$L_n(A) = \sum_{N=1}^{\infty} \frac{1}{N!} \int_{A_{n,N}} dx_1 \dots dx_N$$

For $\zeta \in \Omega$ and $n \in \mathbb{N}$ define

$$\Omega_{n,\zeta} = \{\omega \in \Omega : \omega \subset \Lambda_n, \omega \cup (\zeta \setminus \Lambda_n) \in \Omega\}$$

It is easily seen that $\Omega_{n,\zeta} \in \mathcal{F}$. For $\zeta \in \Omega$, $z \in \mathbb{R}$, and $n \in \mathbb{N}$, define the **grand canonical Gibbs distribution** $G_{n,z,\zeta}$ **with boundary condition** ζ **on** Λ_n by

$$\begin{aligned} G_{n,z,\zeta}(A) &= \Xi_{n,z,\zeta}^{-1} \int_{A \cap \Omega_{n,\zeta}} z^{\#\omega} L_n(d\omega) \\ \Xi_{n,z,\zeta} &= \int_{\Omega_{n,\zeta}} z^{\#\omega} L_n(d\omega) \end{aligned} \tag{1}$$

for $A \in \mathcal{F}$. The Gibbs distribution $G_{n,z,\zeta}$ is a probability measure on (Ω, \mathcal{F}) with support in $\Omega_{n,\zeta}$. A measure μ_z on (Ω, \mathcal{F}) is called a **Gibbs measure** if $\mu_z(\Omega) = 1$ and for all $n \in \mathbb{N}$ and all measurable functions $f : \Omega \rightarrow [0, \infty)$,

$$\int_{\Omega} f(\omega) \mu_z(d\omega) = \int_{\Omega} \mu_z(d\zeta) \int_{\Omega_{n,\zeta}} G_{n,z,\zeta}(d\omega) f(\omega \cup (\zeta \setminus \Lambda_n)) \tag{2}$$

It is well known that μ_z exists for every z . (For a proof of existence, see [5].) However, μ_z may be non-unique. When μ_z is referred to below, it is assumed μ_z is an arbitrary Gibbs measure.

For $s > 0$, $P, Q \subset \mathbb{R}^2$ and $x \in \mathbb{R}^2$, define

$$\begin{aligned} B_s(x) &= \{y \in \mathbb{R}^2 : |x - y| \leq s\} \\ d(P, Q) &= \inf\{|p - q| : p \in P, q \in Q\} \\ P - x &= \{p - x : p \in P\} \end{aligned}$$

and call P **infinite** if for every n , P is not a subset of Λ_n . The main result of this paper (Theorem 5.2) is the following: Let $L > 3r$, and let A_{inf} be the event that $\cup_{x \in \omega} B_{L/2}(x)$ has an infinite connected component, W , such that $d(0, W) \leq L/2$. Then $A_{inf} \in \mathcal{F}$ and $\lim_{z \rightarrow \infty} \mu_z(A_{inf}) = 1$.

Here an outline of the proof is sketched. Write $R = \delta + 3r/2$ with $\delta > 0$, and let $\Psi : \mathbb{R}^2 \rightarrow (\epsilon\mathbb{Z})^2$ be a discretization of space such that $\epsilon \ll r$. Let $\omega \in \Omega$, and suppose $\cup_{x \in \omega} B_R(\Psi(x))$ has a finite connected component W . The boundary of W is comprised of a number of closed curves; let γ be the one which encloses a region W_γ containing all the others, and assume γ is comprised of exactly K arcs. Let A_γ be the set of all $\omega \in \Omega$ for which the curve γ arises as above. It can be shown that for ϵ sufficiently small (depending only on δ), there is a vector $u_0 \in \mathbb{R}^2$ of magnitude $\sim r$ and an L_n -preserving map $\phi : A_\gamma \rightarrow \Omega$ defined by

$$\phi(\omega) = ((\omega \cap W_\gamma) - u_0) \cup (\omega \setminus W_\gamma)$$

with the following property: that there exist $x_1, x_2, \dots, x_M \in \mathbb{R}^2$, with $M = \lceil cK \rceil$ and c a positive constant, such that for all $\omega \in A_\gamma$ and $i \neq j \in \{1, 2, \dots, M\}$,

$$\begin{aligned} d(x_i, \phi(\omega)) &\geq \delta/2 + 2r \\ |x_i - x_j| &\geq \delta + 2r \end{aligned}$$

(Here c depends only on δ and r .) With

$$A_\gamma^\phi = \{\phi(\omega) \cup \{y_1, y_2, \dots, y_M\} : \omega \in A_\gamma, y_i \in B_{\delta/2}(x_i)\}$$

from (2) it can be shown that

$$G_{n,z,\zeta}(A_\gamma) \leq \frac{G_{n,z,\zeta}(A_\gamma)}{G_{n,z,\zeta}(A_\gamma^\phi)} = (\pi\delta^2 z/4)^{-M}$$

provided n is large enough. It follows that

$$\mu_z(A_\gamma) \leq (\pi\delta^2 z/4)^{-M}$$

Now let A_{inf}^Ψ be the event that $\cup_{x \in \omega} B_R(\Psi(x))$ has an infinite connected component W such that $d(0, W) \leq L/2$. (It is easy to see that $A_{inf}^\Psi \in \mathcal{F}$.) Consider only those finite connected components W of $\cup_{x \in \omega} B_R(\Psi(x))$ such that $d(0, W) \leq L/2$. A counting argument shows that the number of curves γ with K arcs corresponding to such W is bounded above by

$$((K+1)H/\epsilon)^2 (H/\epsilon)^{2(K-1)}$$

where H depends only on r and L . So the μ_z -probability that there is a finite connected component W of $\cup_{x \in \omega} B_R(\Psi(x))$ such that $d(0, W) \leq L/2$ is less than

$$\sum_{K=1}^{\infty} ((K+1)H/\epsilon)^2 (H/\epsilon)^{2(K-1)} (\pi\delta^2 z/4)^{-\lceil cK \rceil}$$

This summation approaches zero as $z \rightarrow \infty$. On the other hand, the μ_z -probability that $d(0, W) > L/2$ for all connected components W of $\cup_{x \in \omega} B_R(\Psi(x))$ also approaches zero as $z \rightarrow \infty$. Thus,

$$\mu_z(A_{inf}^\Psi) \rightarrow 1 \quad \text{as } z \rightarrow \infty$$

The continuous space corollary is the statement $\lim_{z \rightarrow \infty} \mu_z(A_{inf}) = 1$. This immediately follows so long as $A_{inf} \in \mathcal{F}$. In fact it will be shown that A_{inf} is a closed set in the topology \mathcal{T} defined above.

3. DISCRETIZATION AND CONTOURS

Fix $R = \delta + 3r/2$ with $\delta \in (0, r/2)$. Let $\epsilon \in (0, r/2)$, and define $\Psi : \mathbb{R}^2 \rightarrow (\epsilon\mathbb{Z})^2$ as follows. If for $n, m \in \mathbb{Z}$,

$$(x, y) \in [\epsilon m - \epsilon/2, \epsilon m + \epsilon/2) \times [\epsilon n - \epsilon/2, \epsilon n + \epsilon/2)$$

then set

$$\Psi(x, y) = (\epsilon m, \epsilon n)$$

Note that $|\Psi(x) - x| < \epsilon$ for all $x \in \mathbb{R}^2$. Furthermore Ψ is Borel measurable in the sense that $\Psi^{-1}(P)$ is a Borel set for any $P \subset (\epsilon\mathbb{Z})^2$. (The dependence of Ψ on ϵ will be suppressed.)

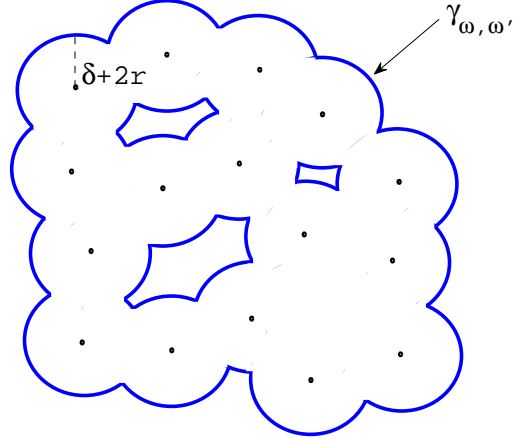


FIGURE 1. The outer curve is a contour $\gamma = \gamma_{\omega, \omega'}$ of size 13. All the points pictured belong to $\Psi(\omega')$.

Let $\omega \in \Omega$. The connected components of $\cup_{x \in \omega} B_R(\Psi(x))$ naturally partition ω into subsets $\omega' \subset \omega$; each ω' consists exactly of all the points $x \in \omega$ such that $\Psi(x)$ belongs to a given connected component of $\cup_{x \in \omega} B_R(\Psi(x))$. The subsets ω' will be called **components** of ω . A component ω' of ω is said to be **finite** if $\omega' \subset \Lambda_n$ for some n . For each finite component ω' of $\omega \in \Omega$, consider the set

$$W_{\omega, \omega'} = \cup_{x \in \omega'} B_{\delta+2r}(\Psi(x))$$

Since $\epsilon \in (0, r/2)$, $W_{\omega, \omega'}$ is connected. (It will also be assumed throughout that $r, \delta \in \mathbb{Q}$ and that ϵ is transcendental. This assumption implies that if two disks in $W_{\omega, \omega'}$ intersect, then they overlap.) Consider now the boundary $\partial W_{\omega, \omega'}$ of $W_{\omega, \omega'}$. By the above, $\partial W_{\omega, \omega'}$ is a union of (images of) simple closed curves, one of which encloses a region containing all the others. Define $\gamma = \gamma_{\omega, \omega'} \subset \mathbb{R}^2$ to be the latter curve; γ will be called a **contour** of ω . A contour γ is (the image of) a simple closed curve comprised of circle arcs. The total number of circle arcs in γ is called the **size of the contour**. See Figure 1. **The region enclosed by** γ will be denoted W_γ . It is emphasized that a contour $\gamma = \gamma_{\omega, \omega'}$ is defined only when ω' is a finite component of some $\omega \in \Omega$.

Lemma 3.1. *Let $\epsilon \in (0, \delta/2)$. Then there exists $c > 0$ such that the following holds. Let γ be any contour of size $K > 0$, and let A_γ be the (nonempty) set of all $\omega \in \Omega$ such that $\gamma = \gamma_{\omega, \omega'}$ for some finite component ω' of ω . Then $A_\gamma \in \mathcal{F}$. Choose n such that $\gamma \subset \Lambda_n$. There is a map $\phi : A_\gamma \rightarrow \Omega$ and $x_1, x_2, \dots, x_M \in \mathbb{R}^2$, with $M = \lceil cK \rceil$, such that:*

- (i) $L_n(A_\gamma) = L_n(\phi(A_\gamma))$
- (ii) $|x_i - x_j| \geq \delta + 2r$ for all $i \neq j \in \{1, 2, \dots, M\}$
- (iii) $d(x_i, \phi(\omega)) \geq \delta/2 + 2r$ for all $i \in \{1, 2, \dots, M\}$ and all $\omega \in A_\gamma$

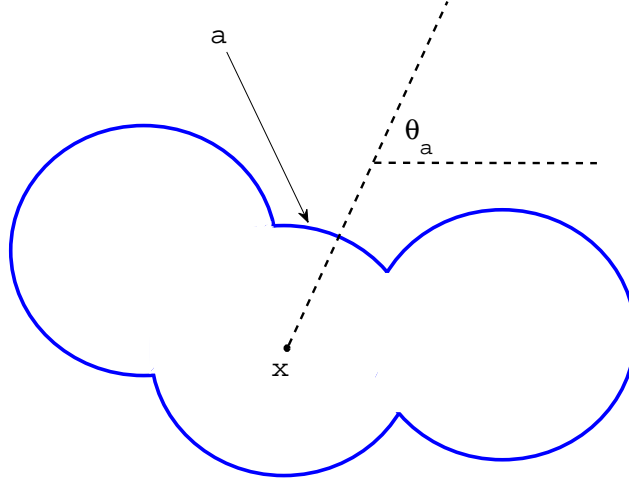


FIGURE 2. A contour $\gamma_{\omega, \omega'}$ with the arc a . θ_a is the outward normal angle with respect to the midpoint of a . Here $x \in \Psi(\omega')$.

Proof. To see that $A_\gamma \in \mathcal{F}$, note that A_γ can be written as a finite intersection of sets of the form $\{\omega \in \Omega : \#(\omega \cap \Psi^{-1}(\{x\})) = \ell\}$, where $x \in (\epsilon\mathbb{Z})^2$ and $\ell \in \{0, 1\}$.

For each circle arc a of γ , let $\theta_a \in [0, 2\pi)$ be an outward normal angle with respect to the midpoint of the arc (see Figure 2). Let $0 < \alpha < \cos^{-1}(1 - \delta/(8r))$ be such that $\alpha = 2\pi/n$ for some $n \in \mathbb{N}$. By the pigeonhole principle, there is a subinterval $I = [v, v + \alpha) \subset [0, 2\pi)$ such that $\lceil (2\pi)^{-1} \alpha K \rceil$ of the angles θ_a belong to I . Fix $\theta_0 = \theta_{a_0} \in I$ corresponding to some arc a_0 of γ , and let $u_0 \in \mathbb{R}^2$ be a vector in the direction of θ_0 with magnitude $\delta/2 + r$ (see Figure 3). Define $\phi : \mathcal{P}(\mathbb{R}^2) \rightarrow \mathcal{P}(\mathbb{R}^2)$ by

$$\phi(X) = ((X \cap W_\gamma) - u_0) \cup (X \setminus W_\gamma)$$

It will be shown below that $\phi(A_\gamma) \subset \Omega$. Let $\omega \in A_\gamma$ be arbitrary, and let ω' be the unique component of ω such that $\gamma = \gamma_{\omega, \omega'}$.

Assume $x \in \omega \setminus W_\gamma$. Then $d(\Psi(x), \Psi(\omega')) > 2\delta + 3r$, and so

$$d(\Psi(x), \cup_{y \in \omega'} B_{\delta+2r}(\Psi(y))) > \delta + r$$

It follows that $d(\Psi(x), \gamma) > \delta + r$, so that $d(x, \gamma) > \delta/2 + r$. Now assume $x \in \omega \cap W_\gamma$. If $x \in \omega'$ then $d(\Psi(x), \gamma) \geq \delta + 2r$, and so $d(x, \gamma) > \delta/2 + 2r$. If $x \notin \omega'$ then

$$\Psi(x) \notin \cup_{y \in \omega'} B_{2\delta+3r}(\Psi(y))$$

and a simple computation shows $d(\Psi(x), \gamma) > \sqrt{5r^2 + 8r\delta + 3\delta^2} > \delta + 2r$, so that $d(x, \gamma) > \delta/2 + 2r$. (See Figure 4).

Define

$$\begin{aligned} A_\gamma^{in} &= \{\omega \cap W_\gamma : \omega \in A_\gamma\} \\ A_\gamma^{out} &= \{\omega \setminus W_\gamma : \omega \in A_\gamma\} \end{aligned}$$

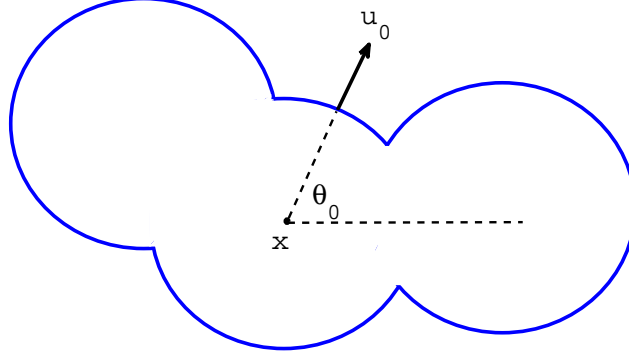


FIGURE 3. The angle $\theta_0 \in I$ and $u_0 = ((r + \delta/2) \cos \theta_0, (r + \delta/2) \sin \theta_0)$.

The preceding paragraph shows that

$$\begin{aligned}\omega^{out} \in A_\gamma^{out} &\Rightarrow d(\omega^{out}, \gamma) > \delta/2 + r \\ \omega^{in} \in A_\gamma^{in} &\Rightarrow d(\omega^{in}, \gamma) > \delta/2 + 2r\end{aligned}$$

Now, let $\omega^{in} \in A_\gamma^{in}$ and $\omega^{out} \in A_\gamma^{out}$, and let $x \in \omega^{in}$, $y \in \omega^{out}$. Let z be any point on the intersection of γ with the line segment \overline{xy} . Then

$$|x - y| = |x - z| + |y - z| > \delta/2 + 2r + \delta/2 + r = \delta + 3r$$

As $|u_0| = \delta/2 + r$, it follows that

$$|\phi(x) - \phi(y)| = |(x - u_0) - y| > \delta/2 + 2r$$

By the preceding statements

$$\begin{aligned}d(\omega^{in}, \omega^{out}) &> \delta + 3r \geq 2r \\ d(\phi(\omega^{in}), \phi(\omega^{out})) &> \delta/2 + 2r \geq 2r\end{aligned}$$

In particular, this implies $\phi(A_\gamma) \subset \Omega$. Also note that $d(\omega^{in}, \gamma) > \delta/2 + 2r$ and $\gamma \subset \Lambda_n$ together imply $\phi(\omega^{in}) = \omega^{in} - u_0 \subset \Lambda_n$. Now

$$\begin{aligned}L_n(A_\gamma) &= L_n(A_\gamma^{in}) L_n(A_\gamma^{out}) \\ &= L_n(A_\gamma^{in} - u_0) L_n(A_\gamma^{out}) \\ &= L_n(\phi(A_\gamma^{in})) L_n(\phi(A_\gamma^{out})) \\ &= L_n(\phi(A_\gamma))\end{aligned}$$

This proves (i).

Consider now (ii) and (iii). Again let $\omega \in A_\gamma$, and let ω' be the unique component of ω such that $\gamma = \gamma_{\omega, \omega'}$. Let m_0 be the midpoint of the arc a_0 of γ (defined above), and let x_0 be the center of the circle (of radius $\delta + 2r$) which forms the arc. As $x_0 \in \Psi(\omega')$, no points of $\Psi(\omega \setminus W_\gamma)$ are in $B_{2\delta+3r}(x_0)$. By definition of u_0 , it follows that for any $x \in \omega \setminus W_\gamma$, $|\Psi(x) - (m_0 - u_0)| > 3\delta/2 + 2r$. (See Figure 5.) Now let m_a be the midpoint of an arc a of γ such that $\theta_a \in I$. By choice of α , an inequality slightly weaker than the preceding one can be obtained: for each $x \in \omega \setminus W_\gamma$, $|\Psi(x) - (m_a - u_0)| > \delta + 2r$. Therefore if $x \in \omega \setminus W_\gamma$ then

$$|\phi(x) - (m_a - u_0)| = |x - (m_a - u_0)| > \delta/2 + 2r$$

On the other hand if $x \in \omega \cap W_\gamma$ then $d(\Psi(x), \gamma) \geq \delta + 2r$, and so

$$|\phi(x) - (m_a - u_0)| = |x - m_a| > \delta/2 + 2r$$

Combining the above statements, if $x \in \omega$ then $|\phi(x) - (m_a - u_0)| > \delta/2 + 2r$.

Now note that for any $x \in \Psi(\omega')$, a disk $B_{2r+\delta}(x)$ contributes to no more than 6 distinct circle arcs in γ . In turn, each circle arc corresponds to a unique $x \in \Psi(\omega')$ which is the center of the circle forming the arc. If two arc midpoints in γ are at distance less than $\delta + 2r$ from one another, then the corresponding $x, y \in \Psi(\omega')$ are at distance less than $3\delta + 6r$, so that the (unique) points in ω' which Ψ maps to x and y are at distance less than $4\delta + 6r < 8r$ from each other. The number of points $x \in \omega$ contained in a disk of radius $8r$ is bounded above by $(8r)^2/r^2 = 64$. The preceding shows that, given any arc midpoint m_a in γ , the number of arc midpoints $m_{\bar{a}} \neq m_a$ in γ such that $|m_a - m_{\bar{a}}| < \delta + 2r$ is bounded above by $J = 6 \cdot 64 = 384$. So with $c = (2\pi(J + 1))^{-1}\alpha$, there exists a subcollection

$$\{m_1, m_2, \dots, m_M\} \subset \{m_a : \theta_a \in I\}, \quad M = \lceil cK \rceil$$

of arc midpoints such that $d(m_i, m_j) \geq \delta + 2r$ for all $i \neq j \in \{1, 2, \dots, M\}$. By taking $x_i = m_i - u_0$ for $i \in \{1, 2, \dots, M\}$, the proof is completed. \square

4. ESTIMATES

Below it is shown that the μ_z -probability of seeing a given contour γ is exponentially small in the size, K , of the contour. This is proved by using Lemma 3.1 to compare each $\omega \in \Omega$ having the contour γ with $\omega^\phi \in \Omega$ of the form $\omega^\phi = \phi(\omega) \cup \{y_1, y_2, \dots, y_M\}$, where $y_i \in B_{\delta/2}(x_i)$. (See Figure 6.)

Lemma 4.1. *Let $\epsilon \in (0, \delta/2)$. Then there exists $c > 0$ such that the following holds. Let γ be any contour of size K , and let A_γ be the set of all $\omega \in \Omega$ such that $\gamma = \gamma_{\omega, \omega'}$ for some finite component ω' of ω . Then*

$$\mu_z(A_\gamma) \leq (\pi\delta^2 z/4)^{-\lceil cK \rceil}$$

Proof. Recall from Lemma 3.1 that $A_\gamma \in \mathcal{F}$. Choose $c > 0$, ϕ and x_1, x_2, \dots, x_M such that the conclusion of Lemma 3.1 is satisfied. Choose \hat{n} so that $\gamma \subset \Lambda_{\hat{n}}$, and let $\zeta \in \Omega$

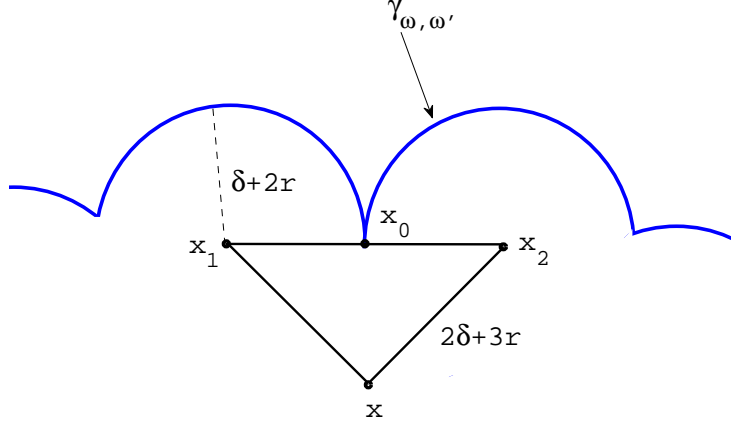


FIGURE 4. Pictured are $x_1, x_2 \in \Psi(\omega') \subset W_\gamma$, and $x \in \Psi(\omega \cap W_\gamma)$ but $x \notin \Psi(\omega')$. For such x , $d(\Psi(x), \gamma) > \sqrt{5r^2 + 8r\delta + 3\delta^2}$. This can be seen in the above picture, in which the distance from x to γ is minimized by placing x_1 and x_2 as far apart as possible.

be arbitrary. For each $A \subset A_\gamma$ such that $A \in \mathcal{F}$, define

$$A^\phi = \{\omega^\phi \subset \mathbb{R}^2 : \omega^\phi = \phi(\omega) \cup \{y_1, y_2, \dots, y_M\}, \omega \in A, y_i \in B_{\delta/2}(x_i)\} \quad (3)$$

By conditions (ii)-(iii) of Lemma 3.1, $A_\gamma^\phi \subset \Omega$. It is easy to see that if $A \subset A_\gamma$ and $A \in \mathcal{F}$, then $A^\phi \in \mathcal{F}$.

By definition of ϕ and choice of \hat{n} , if $\omega \in A_\gamma$ and $\omega^\phi = \phi(\omega) \cup \{y_1, y_2, \dots, y_M\}$, $y_i \in B_{\delta/2}(x_i)$, then $\omega \setminus \Lambda_{\hat{n}+l} = \omega^\phi \setminus \Lambda_{\hat{n}+l}$, where $l = \lceil \delta + r \rceil$. Now let $n = \hat{n} + l + \lceil 2r \rceil$. If $\omega \in A_\gamma$ and $\omega^\phi = \phi(\omega) \cup \{y_1, y_2, \dots, y_M\}$, $y_i \in B_{\delta/2}(x_i)$, then $\omega \in \Omega_{n,\zeta}$ if and only if $\omega^\phi \in \Omega_{n,\zeta}$. Let $A_{\gamma,n,\zeta} = A_\gamma \cap \Omega_{n,\zeta}$. The preceding shows that $A_{\gamma,n,\zeta}^\phi = A_\gamma^\phi \cap \Omega_{n,\zeta}$.

Now, since each disk $B_{\delta/2}(x_i)$ has (Lebesgue) area $\pi\delta^2/4$, choice of n and properties (i)-(iii) of Lemma 3.1 imply

$$L_n(A_{\gamma,n,\zeta}^\phi) = L_n(A_{\gamma,n,\zeta})(\pi\delta^2/4)^M$$

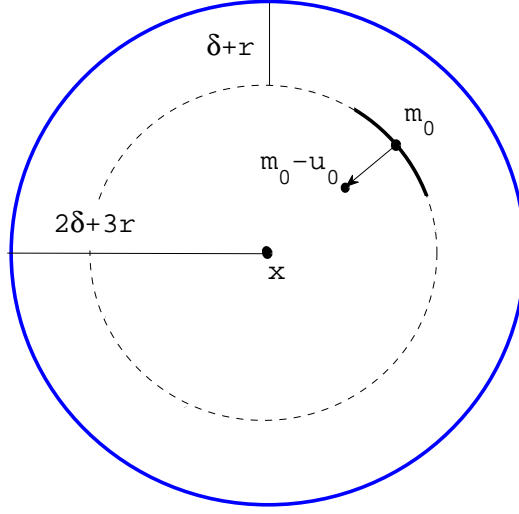


FIGURE 5. The midpoint m_0 of the arc a_0 with corresponding normal vector u_0 . Here $x \in \Psi(\omega')$. No points in $\Psi(\omega \setminus W_\gamma)$ can be inside the large circle. The magnitude of u_0 is $r + \delta/2$, and so the d -distance between $m_0 - u_0$ and the large circle is $3\delta/2 + 2r$.

From definitions it is easy to see that $G_{n,z,\zeta}(A_\gamma)$ and $G_{n,z,\zeta}(A_\gamma^\phi)$ are positive. Comparing the above with (1),

$$\begin{aligned}
G_{n,z,\zeta}(A_\gamma) &\leq \frac{G_{n,z,\zeta}(A_\gamma)}{G_{n,z,\zeta}(A_\gamma^\phi)} \\
&= \frac{\int_{A_\gamma \cap \Omega_{n,\zeta}} z^{\#\omega} L_n(d\omega)}{\int_{A_\gamma^\phi \cap \Omega_{n,\zeta}} z^{\#\omega} L_n(d\omega)} \\
&= z^{-M} \frac{\int_{A_{\gamma,n,\zeta}} L_n(d\omega)}{\int_{A_{\gamma,n,\zeta}^\phi} L_n(d\omega)} \\
&= z^{-M} \frac{L_n(A_{\gamma,n,\zeta})}{L_n(A_{\gamma,n,\zeta}^\phi)} \\
&= (\pi\delta^2 z/4)^{-M}
\end{aligned}$$

Also by choice of n , if $\omega \in \Omega_{n,\zeta}$, then

$$\chi_{A_\gamma}(\omega) = \chi_{A_\gamma}(\omega \cup (\zeta \setminus \Lambda_n))$$

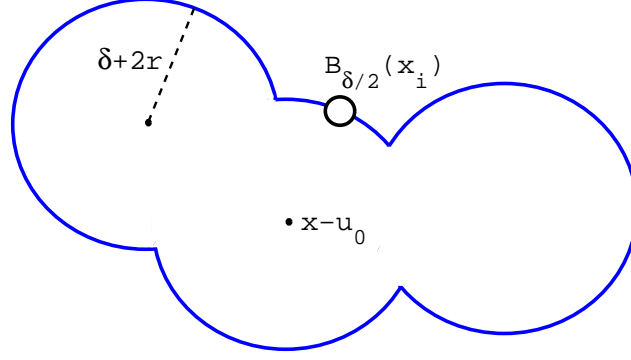


FIGURE 6. A disk $B_{\delta/2}(x_i)$ centered at a midpoint of an arc of $\gamma_{\omega, \omega'} - u_0$, with $x \in \Psi(\omega')$.

where $\chi_{A_\gamma} : \Omega \rightarrow [0, \infty)$ is the (measurable) function

$$\chi_{A_\gamma}(\omega) = \begin{cases} 1, & \omega \in A_\gamma \\ 0, & \omega \notin A_\gamma \end{cases}$$

Since ζ was arbitrary, using (2) with the above gives

$$\begin{aligned} \mu_z(A_\gamma) &= \int_{\Omega} \mu(d\zeta) \int_{\Omega_{n,\zeta}} G_{n,z,\zeta}(d\omega) \chi_{A_\gamma}(\omega \cup (\zeta \setminus \Lambda_n)) \\ &= \int_{\Omega} \mu(d\zeta) \int_{\Omega_{n,\zeta}} G_{n,z,\zeta}(d\omega) \chi_{A_\gamma}(\omega) \\ &= \int_{\Omega} G_{n,z,\zeta}(A_\gamma) \mu(d\zeta) \\ &\leq \int_{\Omega} (\pi\delta^2 z/4)^{-M} \mu(d\zeta) \\ &= (\pi\delta^2 z/4)^{-M} \end{aligned}$$

as desired. □

The number of possible contours passing through a given point is bounded by the number of possible paths along its arcs which start at the point. This number depends on ϵ and the size of the contour. If attention is restricted to contours of a given size which enclose the origin, then a point in the contour must intersect a large ball around the origin. This leads to the following.

Lemma 4.2. *Let $\epsilon \in (0, \delta/2)$, and let Γ_K be the set of all contours γ of size K such that $0 \in W_\gamma$. Then*

$$\#\Gamma_K \leq \left(\frac{(K+1)H}{\epsilon} \right)^2 \left(\frac{H}{\epsilon} \right)^{2(K-1)}$$

where H is a constant depending only on r .

Proof. Note that each contour γ is completely determined by its set of arcs, with each arc naturally corresponding to a unique point in $(\epsilon\mathbb{Z})^2$, namely, the center of the circle of which the arc is part. Let $\gamma \in \Gamma_K$. Since γ is the (image of a) simple closed curve comprised of circle arcs, there is a sequence of circle arcs a_1, a_2, \dots, a_K such that a_i and a_{i+1} are adjacent for $i = 1, 2, \dots, K-1$. Choose the corresponding sequence x_1, x_2, \dots, x_K of points in $(\epsilon\mathbb{Z})^2$. Then

$$|x_{i+1} - x_i| < 2\delta + 4r < 5r \quad (4)$$

for $i = 1, 2, \dots, K-1$.

The number of points in $(\epsilon\mathbb{Z})^2$ inside any disk $B_{sr}(x)$ is bounded above by $2\pi(sr/\epsilon)^2$, provided $s > \sqrt{2}/2$ (see [12]; here the assumption $\epsilon \in (0, \delta/2)$ has been used.) As γ encloses the origin, x_1 must be contained in a disk of radius $(K+1)5r$ around 0. Therefore there are at most

$$\frac{2\pi[(K+1)5r]^2}{\epsilon^2}$$

possibilities for x_1 . Since x_{i+1} must be contained in a disk of radius $5r$ around x_i , given x_i , $i = 1, 2, \dots, K-1$, there are no more than

$$\frac{2\pi(5r)^2}{\epsilon^2}$$

possibilities for x_{i+1} . Taking $H = 5\sqrt{2\pi}r$, the result follows. \square

5. MAIN RESULTS

Let $\omega \in \Omega$. If the origin is not close to an infinite component of ω , then it is either close to a finite component of ω , or it is not close to any component of ω . The probability of the former event can be handled by combining Lemma 4.1 with Lemma 4.2, while it is easy to control the probability of the latter event.

Theorem 5.1. *Let $\epsilon \in (0, \delta/2)$. Let A_{inf}^Ψ be the set of all $\omega \in \Omega$ such that $d(0, \Psi(\omega')) \leq \delta + 2r$ for some infinite component ω' of ω . Then $A_{inf}^\Psi \in \mathcal{F}$, and $\lim_{z \rightarrow \infty} \mu_z(A_{inf}^\Psi) = 1$.*

Proof. Define

$$A_{orig} = \{\omega \in \Omega : d(0, \Psi(\omega')) > \delta + 2r \text{ for all components } \omega' \text{ of } \omega\}$$

$$A_{fin} = \{\omega \in \Omega : d(0, \Psi(\omega')) \leq \delta + 2r \text{ for some finite component } \omega' \text{ of } \omega\}$$

$$A_{cont} = \{\omega \in \Omega : 0 \in W_\gamma \text{ for some contour } \gamma = \gamma_{\omega, \omega'}\}$$

Note that $A_{orig} = \{\omega \in \Omega : \#(\omega \cap \Psi^{-1}(X)) = 0\}$, where $X = \{x \in (\epsilon\mathbb{Z})^2 : |x| \leq \delta + 2r\}$. So $A_{orig} \in \mathcal{F}$. Also, A_{fin}, A_{cont} can each be written as a countable union of finite

intersections of sets of the form $\{\omega \in \Omega : \#(\omega \cap \Psi^{-1}(\{x\})) = \ell\}$ where $x \in (\epsilon\mathbb{Z})^2$ and $\ell \in \{0, 1\}$. Thus $A_{fin}, A_{cont} \in \mathcal{F}$.

Let A_n be the set of all $\omega \in \Omega$ with the following property: that there exist a positive integer k and $x_1, x_2, \dots, x_k \in \Psi(\omega)$ such that $|x_1| \leq \delta + 2r$, $|x_i - x_{i+1}| \leq 2R$ for $i = 1, 2, \dots, k-1$, and $x_k \notin \Lambda_n$. Note that A_n can be written as a finite union of finite intersections of sets of the form $\{\omega \in \Omega : \#(\omega \cap \Psi^{-1}(\{x\})) = 1\}$ where $x \in (\epsilon\mathbb{Z})^2$. Thus, $A_n \in \mathcal{F}$. Since $A_{inf}^\Psi = \bigcap_{n=1}^\infty A_n$, it follows that $A_{inf}^\Psi \in \mathcal{F}$. Now note that

$$\Omega \setminus A_{inf}^\Psi \subset A_{orig} \cup A_{fin}$$

and

$$A_{fin} \subset A_{cont}$$

and so

$$\begin{aligned} \mu_z(\Omega \setminus A_{inf}^\Psi) &\leq \mu_z(A_{orig}) + \mu_z(A_{fin}) \\ &\leq \mu_z(A_{orig}) + \mu_z(A_{cont}) \end{aligned}$$

Choose $c > 0$ such that the conclusion of Lemma 4.1 holds, and choose H such that the conclusion of Lemma 4.2 holds. Then

$$\begin{aligned} \mu_z(A_{cont}) &\leq \sum_{K=1}^\infty \# \Gamma_K (\pi\delta^2 z/4)^{-\lceil cK \rceil} \\ &\leq \sum_{K=1}^\infty \left(\frac{(K+1)H}{\epsilon} \right)^2 \left(\frac{H}{\epsilon} \right)^{2(K-1)} (\pi\delta^2 z/4)^{-\lceil cK \rceil} \end{aligned}$$

This shows that $\mu_z(A_{cont}) \rightarrow 0$ as $z \rightarrow \infty$.

Now for any $\omega \in A_{orig}$, $d(0, \Psi(\omega)) > \delta + 2r$, and so $d(0, \omega) > \delta/2 + 2r$. It follows that for any $\omega \in A_{orig}$ and any $x \in B_{\delta/2}(0)$, $\omega \cup x \in \Omega$. A simplified version of the proof of Lemma 4.1 then implies that

$$\mu_z(A_{orig}) \leq (\pi\delta^2 z/4)^{-1}$$

This shows that $\mu_z(A_{orig}) \rightarrow 0$ as $z \rightarrow \infty$. The result follows. \square

In order to extend Theorem 5.1 to continuous space, it must be proved that the relevant percolation event is measurable. This is the main content of Theorem 5.2 below. The following notation will be needed. Let $\omega \in \Omega$ and $P, Q \subset \mathbb{R}^2$ be Borel sets, and let $L > 0$. It will be said that **there is an L -path in ω from P to Q** if (for some positive integer k) there exist $x_1, x_2, \dots, x_k \in \omega$ such that $x_1 \in P$, $x_k \in Q$, and $|x_i - x_{i+1}| \leq L$ for $i = 1, 2, \dots, k-1$. Also let

$$B_s^{int}(x) = \{y \in \mathbb{R}^2 : |y - x| < s\}$$

Theorem 5.2. *Let $L > 3r$. Let A_{inf} be the set of all $\omega \in \Omega$ such that $\bigcup_{x \in \omega} B_{L/2}(x)$ has an infinite connected component, W , such that $d(0, W) \leq L/2$. Then $A_{inf} \in \mathcal{F}$, and $\lim_{z \rightarrow \infty} \mu_z(A_{inf}) = 1$.*

Proof. Define

$$A_n = \{\omega \in \Omega : \text{there is no } L\text{-path in } \omega \text{ from } B_L(0) \text{ to } \mathbb{R}^2 \setminus (-n, n)^2\}$$

Fix $\omega_0 \in A_n$. It will be shown that there exists $A \in \mathcal{T}$ such that $\omega_0 \in A \subset A_n$. As $\{\emptyset\} \in \mathcal{T}$ it may be assumed that $\omega_0 \neq \emptyset$. Let

$$L_0 = \inf\{L' : \text{there is an } L'\text{-path in } \omega_0 \text{ from } B_L(0) \text{ to } \mathbb{R}^2 \setminus (-n, n)^2\}$$

with $L_0 = \infty$ if $\omega_0 \cap B_L(0) = \emptyset$ or $\omega_0 \cap (\mathbb{R}^2 \setminus (-n, n)^2) = \emptyset$. Since $\omega_0 \in A_n$, $L_0 > L$. Choose $L_1 \in (L, L_0)$ and let

$$\eta = \inf\{\zeta : \omega_0 \text{ there is an } L_1\text{-path from } B_{L+\zeta}(0) \text{ to } \mathbb{R}^2 \setminus (-n + \zeta, n - \zeta)^2\}$$

with $\eta = \infty$ if $\omega_0 \cap B_{L+n}(0) = \emptyset$. Since $L_1 < L_0$, $\eta > 0$. Choose $\beta > 0$ such that

$$\beta < \min\{r, \eta, (L_1 - L)/2\}$$

and $\lambda > n + L$ such that

$$\begin{aligned} d(\omega_0, \partial[-\lambda, \lambda]^2) &> 0 \\ \omega_0 \cap [-\lambda, \lambda]^2 &\neq \emptyset \end{aligned}$$

Set

$$Y = \cup_{x \in \omega_0} B_\beta^{int}(x) \cap (-\lambda, \lambda)^2$$

and

$$A_Y = \{\omega \in \Omega : \#(\omega \cap Y) = \#(\omega \cap [-\lambda, \lambda]^2) = \#(\omega_0 \cap [-\lambda, \lambda]^2)\}$$

Then $A_Y \in \mathcal{T}$, and by construction $\omega_0 \in A_Y$. It will be shown that $A_Y \subset A_n$. Let $\omega \in A_Y$, and assume (for contradiction) that $\omega \notin A_n$. Then there exists an L -path y_1, y_2, \dots, y_l in ω from $B_L(0)$ to $\mathbb{R}^2 \setminus (-n, n)^2$. Let

$$k = \min\{j \in \{1, 2, \dots, l\} : y_j \notin (-n, n)^2\}$$

Then $y_i \in (-\lambda, \lambda)^2$ for $i = 1, 2, \dots, k$. By definition of A_Y , for each $i = 1, 2, \dots, k$, there exists $x_i \in \omega_0$ such that $y_i \in B_\beta^{int}(x_i)$. If $x_1 \notin B_L(0)$ or $x_k \notin \mathbb{R}^2 \setminus (-n, n)^2$, then x_1, x_2, \dots, x_k is a L_1 -path in ω_0 from $B_{L+\beta}(0)$ to $\mathbb{R}^2 \setminus (-n + \beta, n - \beta)^2$, a contradiction since $\beta < \eta$. So assume $x_1 \in B_L(0)$ and $x_k \in \mathbb{R}^2 \setminus (-n, n)^2$. Then x_1, x_2, \dots, x_k is an L_1 -path from $B_L(0)$ to $\mathbb{R}^2 \setminus (-n, n)^2$, a contradiction since $L_1 < L_0$. So $\omega \in A_n$. As $\omega \in A_Y$ was arbitrary, $A_Y \subset A_n$. It has been shown that $A_n \in \mathcal{T} \subset \mathcal{F}$. Thus

$$A_{inf} = \cap_{n=1}^\infty (\Omega \setminus A_n) \in \mathcal{F}$$

Now choose $\delta \in (0, r/2)$ and $\epsilon \in (0, \delta/2)$ such that $3r + 2\delta + 2\epsilon < L$. Define A_{inf}^Ψ as in Theorem 5.1. Then $A_{inf}^\Psi \subset A_{inf}$ and so

$$\mu_z(A_{inf}) \geq \mu_z(A_{inf}^\Psi)$$

The result now follows from Theorem 5.1. □

Theorem 5.2 has the following interpretation. Let $x \in \mathbb{R}$ be arbitrary, and let $\omega \in \Omega$ be sampled from the Gibbs measure μ_z with exclusion radius $2r$. Connect each pair of points in ω by an edge if they are at distance at most $L > 3r$ from one another. Call the resulting graph G . Then if z is sufficiently large, with positive probability G has an infinite connected component close to x . If $L < 4r$, an infinite connected component in G corresponds to an infinite connected region of excluded volume in ω . For completeness the following corollary is included:

Corollary 5.3. *Let $L > 3r$. Let A be the set of all $\omega \in \Omega$ such that $\cup_{x \in \omega} B_{L/2}(x)$ has an infinite connected component. Then $A \in \mathcal{F}$, and $\lim_{z \rightarrow \infty} \mu_z(A) = 1$.*

Proof. Let A_{inf} be defined as in Theorem 5.2. Then $A_{inf} \subset A$, so the result follows from Theorem 5.2 so long as $A \in \mathcal{F}$. Choose $x_1, x_2, \dots \in \mathbb{R}^2$ such that

$$\cup_{i=1}^{\infty} B_{L/2}(x_i) = \mathbb{R}^2$$

Let A_{inf}^z be the set of all $\omega \in \Omega$ such that $\cup_{x \in \omega} B_{L/2}(x)$ has an infinite connected component, W , such that $d(z, W) \leq L/2$. Following the arguments of the proof of Theorem 5.2, $A_{inf}^z \in \mathcal{F}$ for each $z \in \mathbb{R}^2$. Thus

$$A = \cup_{i=1}^{\infty} A_{inf}^{x_i} \in \mathcal{F}$$

as desired. □

It should be noted that in many standard percolation problems, a zero-one law can be used to show that either $\mu_z(A) = 0$ or $\mu_z(A) = 1$ [13]. The zero-one laws do not apply in this setting, however, as the requisite properties of independence do not hold.

6. CONCLUSION

Percolation of excluded volume has been proved for points in the plane distributed according to a Gibbs measure with a pure hard core interaction. This model, commonly called the *hard disk model*, is among the simplest continuum models of particles with pair interactions. The proof, which generalizes to 3D, relies on a Peierls-type argument [14]. (The generalization requires a slightly more complicated argument for estimating the number of contours of a given size; otherwise the proof is exactly the same.) A similar result is expected in a hard disk model with an added attraction which extends beyond the hard core, though this generalization is not pursued here. The hard disk model with attraction is believed to exhibit a gas-liquid phase transition, which has been heuristically connected to percolation of excluded volume [8],[9]. (There is no proof in the literature of a gas-liquid transition in a continuum model with pair interactions; see, however, [15].) To this author's knowledge, there is no previous proof of percolation of hard disks (or spheres) in the literature. (See [7] for a proof in a model with a complicated exclusion.) In general, very little is known (or proved) about the qualitative properties of the hard disk model at large activity. The result of this paper is of particular interest because of the known absence of long range translational order in the model. It remains an open question whether percolation occurs for an arbitrarily small connection radius, that is, for a connection radius extending just beyond the exclusion radius [7].

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